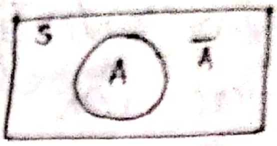
estimators for α_i and β_i, when the model has a single regressor x_i

THEORY OF PROBABILITY

Q. Probability of the complementary event $\bar{A}$ of A is given by
$$P(\bar{A}) = 1 - P(A).$$

A and $\bar{A}$ are mutually disjoint events, so that.

$$A \cup \bar{A} = S$$



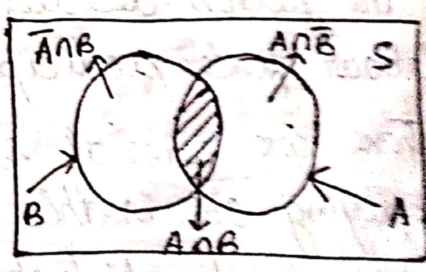
$$\Rightarrow P(A \cup \bar{A}) = P(S)$$

$$\Rightarrow P(A) + P(\bar{A}) = 1$$ [From the axioms of certainty and axioms of additivity]

$$\Rightarrow P(\bar{A}) = 1 - P(A).$$ (Proved)

Q. If A and B are any two events (subsets of sample space S) and are not mutually exclusive (disjoint) then,
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

From the venn diagram we have,



$$A \cup B = A \cup (\bar{A} \cap B)$$

where A and $\bar{A} \cap B$ are mutually exclusive or disjoint.

$$\therefore P(A \cup B) = P[A \cup (\bar{A} \cap B)]$$

$$= P(A) + P(\bar{A} \cap B)$$ [By the axiom of additivity]

$$= P(A) + [P(\bar{A} \cap B) + P(A \cap B)] - P(A \cap B)$$

$$= P(A) + P[(\bar{A} \cap B) \cup (A \cap B)] - P(A \cap B)$$ [$\because$ $\bar{A} \cap B$ and $A \cap B$ are disjoint]

$= P(A) + P(B) - P(A \cap B)$

$\therefore$ we have, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (Proved)

8. For two events, A and B,

$$P(A \cap B) = P(A) \cdot P(B|A), \quad P(A) > 0$$

$$= P(B) \cdot P(A|B), \quad P(B) > 0$$

where $P(B|A)$ represents conditional probability of occurrence of B when the event A has already happened and $P(A|B)$ is the conditional probability of happening of A, given that B has already happened.

In usual notations, we have,

$P(A) = \dfrac{n(A)}{n(S)}$ where, $n(S)$ = no. of sample points in S

$P(B) = \dfrac{n(B)}{n(S)}$ $n(A)$ = no. of sample points in A

$n(B)$ = no. of sample points in B.

$P(A \cap B) = \dfrac{n(A \cap B)}{n(S)}$ $n(A \cap B)$ = no. of sample points in $(A \cap B)$

For the conditional event (A/B), the favourable outcomes must be one of the sample points of B i.e. for the event A/B, the sample space is B and out of the $n(B)$ sample points $n(A \cap B)$ pertain to the occurrence of the event A. Hence,

$$P(A/B) = \frac{n(A \cap B)}{n(B)}.$$

$$\therefore P(A \cap B) = \frac{n(B)}{n(S)} \times \frac{n(A \cap B)}{n(B)}$$

$$\Rightarrow P(A \cap B) = P(B) \cdot P(A/B) \quad \text{——— (1)}$$

Similarly, $P(B/A) = \dfrac{n(A \cap B)}{n(A)}$

$$\therefore P(A \cap B) = \frac{n(A)}{n(S)} \times \frac{n(A \cap B)}{n(A)}$$

$$\Rightarrow P(A \cap B) = P(A) \cdot P(B/A) \quad \text{——— (2)}$$

(1) and (2) gives the required result.

Q. If A and B are two events with positive probabilities $\{P(A)>0,\ P(B)>0\}$, then A and B are independent if and only if, $P(A\cap B) = P(A)\cdot P(B)$.

We have,

$$P(A\cap B) = P(A)\cdot P(B/A) = P(B)\cdot P(A/B)$$

and $P(A) \neq 0$, $P(B) \neq 0$.

If A and B are independent i.e. A is independent of B and B is independent of A, then we have,

$$P(A/B) = P(A) \quad \text{and} \quad P(B/A) = P(B).$$

Therefore,

$$P(A\cap B) = P(A)\cdot P(B).$$ Hence, the theorem is proved.

Q. If X and Y are random variables then,

$$E(X+Y) = E(X) + E(Y)$$

provided all the expectations exist.

Let, X and Y be continuous random variables with joint p.d.f $f_{XY}(x,y)$ and marginal p.d.f's $f_X(x)$ and $f_Y(y)$ respectively. Then, by definition,

$$E(X) = \int_{-\infty}^{\infty} x\cdot f_X(x)\, dx \quad \text{——— (1)}$$

$$E(Y) = \int_{-\infty}^{\infty} y\cdot f_Y(y)\cdot dy \quad \text{——— (2)}$$

$$E(X+Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)\ f_{XY}(x,y)\cdot dx\cdot dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\cdot f_{XY}(x,y)\cdot dx\cdot dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\cdot f_{XY}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x\left[\int_{-\infty}^{\infty} f_{XY}(x,y)\, dy\right] dx + \int_{-\infty}^{\infty} y\left[\int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\right] dy$$

$$= \int x\left[\int f_X(x)\ f_Y(y)\, dy\right]\cdot dx$$

$$= \int_{-\infty}^{\infty} x \cdot f_X(x) \cdot dx + \int_{-\infty}^{\infty} y \cdot f_Y(y)\, dy$$

$= E(X) + E(Y)$ [By using (1) and (2) we get]

$\therefore E(X+Y) = E(X) + E(Y)$ Hence, it is proved.

If X and Y are independent random variables, then,

$$E(XY) = E(X) \cdot E(Y)$$

Let X and Y be continuous random variables and independent with marginal p.d.f's $f_X(x)$ and $f_Y(y)$ respectively. Then by definition,

$$\left.\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x \cdot f_X(x) \cdot dx \\ E(Y) &= \int_{-\infty}^{\infty} y \cdot f_Y(y) \cdot dy \end{aligned}\right\} \text{—— (1)}$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot f_{XY}(x,y) \cdot dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot f_X(x) \cdot f_Y(y) \cdot dx \cdot dy$$ [$\because$ X and Y are independent random variables]

$$= \int_{-\infty}^{\infty} x\, f_X(x)\, dx \int_{-\infty}^{\infty} y \cdot f_Y(y) \cdot dy$$

$= E(X) \cdot E(Y)$ [By using (1)]

$\therefore E(XY) = E(X) \cdot E(Y)$ Hence it is proved.

Q. If x_1 and x_2 are random variables then,

(i) $Var(ax_1 + bx_2) = a^2 V(x_1) + b^2 V(x_2) + 2ab\, Cov(x_1, x_2)$

(ii) $Var(ax_1 - bx_2) = a^2 Var(x_1) + b^2 Var(x_2) - 2ab\, Cov(x_1, x_2)$

Let, $u = ax_1 + bx_2$

$$E(u) = a E(x_1) + b E(x_2)$$

$$\therefore u - E(u) = ax_1 + bx_2 - aE(x_1) - bE(x_2)$$

$$= a[x_1 - E(x_1)] + b[x_2 - E(x_2)]$$

Squaring and taking expectation on both sides we get,

$$E[u - E(u)]^2 = E\left[a[x_1 - E(x_1)] + b[x_2 - E(x_2)]\right]^2$$

$$= E\left[a^2\{x_1 - E(x_1)\}^2 + b^2\{x_2 - E(x_2)\}^2 + 2ab[\{x_1 - E(x_1)\}\{x_2 - E(x_2)\}]\right]$$

$$= a^2 E[x_1 - E(x_1)]^2 + b^2 E[x_2 - E(x_2)]^2 + 2ab\, E[\{x_1 - E(x_1)\}\{x_2 - E(x_2)\}]$$

$$= a^2 Var(x_1) + b^2 Var(x_2) + 2ab\, Cov(x_1, x_2)$$

$$\therefore Var(u) = a^2 Var(x_1) + b^2 Var(x_2) + 2ab\, Cov(x_1, x_2)$$

$$\therefore Var(ax_1 + bx_2) = a^2 Var(x_1) + b^2 Var(x_2) + 2ab\, Cov(x_1, x_2)$$

Similarly,

$$Var(ax_1 - bx_2) = a^2 Var(x_1) + b^2 Var(x_2) - 2ab\, Cov(x_1, x_2)$$

If $a = b = 1$ and x_1 and x_2 are independent random variables, then, $Cov(x_1, x_2) = 0$.

$$Var(x_1 \pm x_2) = Var(x_1) + Var(x_2)$$

$$\therefore Var(x_1) + Var(x_2) = Var(x_1 + x_2) = Var(x_1 - x_2).$$

Q. Define moment generating function (m.g.f). How will you get the moments about origin and moments about mean from m.g.f ?

The moment generating function (m.g.f) of a random variable x (about origin) having the probability function $f(x)$ is given by.

$$M_x(t) = E[e^{tx}] = \begin{cases} \int e^{tx} f(x)\,dx \text{ (for continuous P.D)} \\ \sum_x e^{tx} f(x) \text{ (for discrete P.D)} \end{cases}$$

Thus,

$$M_x(t) = E[e^{tx}]$$

$$= E\left[1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \frac{t^4x^4}{4!} + \cdots + \frac{t^r x^r}{r!} + \cdots\right]$$

$$= 1 + t\,E(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \frac{t^4}{4!}E(x^4) + \cdots + \frac{t^r}{r!}E(x^r)$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \frac{t^4}{4!}\mu_4' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

where, $\mu_r' = E(x^r) = \begin{cases} \int x^r f(x)\,dx \rightarrow \text{for continuous distribution} \\ \sum_x x^r p(x) \rightarrow \text{for discrete distribution} \end{cases}$

is the r^{th} moment of x about origin. Thus, we see that the coefficient of $\frac{t^r}{r!}$ in $M_x(t)$ gives μ_r' (about origin). Since $M_x(t)$ generates moments, it is known as moment generating function.

Now, if we take the moment generating function of x about mean $(\bar{x})$ then we get,

$$M_{(x-\bar{x})}(t) = E\left[e^{t(x-\bar{x})}\right]$$

$$= E\left[1 + \frac{t(x-\bar{x})}{1!} + \frac{t^2(x-\bar{x})^2}{2!} + \frac{t^3(x-\bar{x})^3}{3!} + \frac{t^4(x-\bar{x})^4}{4!} + \cdots + \frac{t^r(x-\bar{x})^r}{r!} + \cdots\right]$$

$$= 1 + t\,E(x-\bar{x}) + \frac{t^2}{2!}E(x-\bar{x})^2 + \frac{t^3}{3!}E(x-\bar{x})^3 + \frac{t^4}{4!}E(x-\bar{x})^4 + \cdots + \frac{t^r}{r!}E(x-\bar{x})^r + \cdots$$

$$= 1 + 0 + \frac{t^2}{2!}\mu_2 + \frac{t^3}{3!}\mu_3 + \frac{t^4}{4!}\mu_4 + \cdots + \frac{t^r}{r!}\mu_r + \cdots$$

This is the moment generating function of x (about mean). which gives us the central moments.

8. State and explain the important properties of moment generating function (m.g.f).

The properties of m.g.f are discussed given below

(1) $M_{cx}(t) = M_x(ct)$. c being constant -

By definition,

$$LHS = M_{cx}(t) = E\left(e^{tcx}\right).$$

$$RHS = M_x(ct) = E\left[e^{tcx}\right] = LHS. \text{ (Proved)}$$

(2) The moment generating function of the sum of a number of independent random variables is equal to the product of their respective moment generating functions.

Symbollically, if $x_1, x_2, \ldots, x_n$ are independent random variables, then the moment generating function of their sum $(x_1 + x_2 + \cdots + x_n)$ is given by.

$$M_{x_1+x_2+\cdots+x_n}(t) = M_{x_1}(t) \cdot M_{x_2}(t) \cdots M_{x_n}(t).$$

By definition,

$$M_{x_1+x_2+\cdots+x_n}(t) = E\left[e^{t(x_1+x_2+\cdots+x_n)}\right]$$

$$= E\left[e^{tx_1} \cdot e^{tx_2} \cdots e^{tx_n}\right]$$

$= E[e^{tx_1}] \cdot E[e^{tx_2}] \cdot E[e^{tx_3}] \cdots E[e^{tx_n}]$

$[\because x_1, x_2, \ldots x_n$ are independent$]$

$= M_{x_1}(t) \cdot M_{x_2}(t) \cdots M_{x_n}(t)$.

$$\therefore M_{x_1+x_2+\cdots+x_n}(t) = \prod_{i=1}^{n} M_{x_i}(t). \quad \text{(Proved)}$$

(3) Effect of change of origin and scale on M.G.F.

Let us transform, x to the new variable u by changing both the origin and scale in x as follows

$$u = \frac{x-a}{b}.$$ where a and b are constants.

M.G.F of u (about origin) is given by.

$$M_u(t) = E[e^{tu}]$$

$$= E\left[e^{t\left(\frac{x-a}{b}\right)}\right]$$

$$= E\left[e^{\frac{tx}{b} - \frac{at}{b}}\right]$$

$$= e^{-at/b} \cdot E\left[e^{tx/b}\right]$$

$$= e^{at/b} \cdot M_x(t/b).$$

where $M_x(t)$ is the m.g.f of x about origin.

In particular, if we take.

$$a = E(x) = \mu \text{ (say) and } b = \sigma_x = \sigma \text{ (say)}.$$

then,

$$u = \frac{x - E(x)}{\sigma_x} = \frac{x-\mu}{\sigma} = z \text{ (say)}$$ is known as a standard variate.

Thus, the m.g.f of a standard variate z is given by.

$$M_Z(t) = e^{-\mu t/\sigma} \cdot M_X(t/\sigma)$$

Here, it is to be noted that,

$$E(Z) = E\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma} E[X-\mu] = 0$$

$$Var(Z) = V\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma^2} V(X-\mu) = \frac{\sigma^2}{\sigma^2} = 1$$

$\therefore$ Mean and variance of a standard variate are zero and one.

(4) Uniqueness theorem of m.g.f — The m.g.f of a function of distribution, if it exists, uniquely determines the distribution. This implies that corresponding to a given probability distribution, there is only one m.g.f (provided it exists), and corresponding to a given m.g.f, there is only one probability distribution.

Hence, $M_X(t) = M_Y(t)$

$\Rightarrow$ X and Y are identically distributed.